NEW SEQUENTIAL PROCEDURES FOR SELECTING THE BEST OF k BINOMIAL POPULATIONS, WITH TABLES AND COMPARISONS[1]

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1. Introduction.

Recently the problem of selecting the best one of several binomial populations has been studied from the point of view of different sampling rules. In this paper we compare some sequential procedures with and without early elimination. The main breakdown is between those using the cyclic-play-the-winner (PWC) sampling rule and those using the vector-at-a-time (VT) sampling rule.

The PWC-rule orders the k given populations at random at the outset and uses this ordering in a cyclic manner. After each success we sample from the same population; after each failure we switch to the next population in the ordering scheme. After the k^{th} population we complete the cycle by going back to the first population.

The VT-rule consists of taking k-tuple observations, one component from each population. In a variation of this, the cyclic (VTC) rule, we start as in the PWC rule by randomizing the order of the populations and then take one observation from each population using the fixed cyclic order; thus we need not complete the last vector in the VTC rule.

Both of the above rules can be modified as follows. Let the order of the populations sampled be $\pi_1, \pi_2, \ldots, \pi_k$. From the beginning of sampling π_1 to the end of sampling π_k, we have gone through one complete sampling cycle. Our new modification is to reorder the k populations after each complete sampling cycle; this reordering <u>can</u> depend on the observed results. We denote such a modification of the PWC and VTC rules by PWO and VTO, respectively.

Several papers dealing with the PW and VT sampling rules [6], [9], [12], and [13] consider termination rules based on a fixed sample size

or on inverse sampling, i.e., we sample until at least one population reaches a fixed number of successes. In [10] $k = 2$ and the termination rule is based on the difference of the numbers of successes. The book [1] deals mainly with VT-sampling and a stopping rule based on likelihood ratios. (A summary of the above work can be found in [11].) Paulson [7], [8] has brought in early elimination techniques, which (except for [1]) is not a feature of the above references; some discussion of elimination procedures does appear in Chapter 9 of [1].

In this paper we introduce a new procedure that combines the likelihood approach to the stopping rule with the PWC sampling rule. In section 4 we derive an extension of the rule in [10] to the case of k populations, which contains the feature of early elimination. Empirical results for the PWO sampling rules are obtained and analyzed for the inverse sampling rule in section 6. Sections 3, and 4 deal with the PW sampling while section 6 is concerned with VT sampling. In section 5 sequential techniques developed in [3] are applied to Wald's sequential double dichotomy formulation [14], producing a binomial selection procedure with VT sampling and early elimination features.

In section 7, we briefly describe two other VT rules, originally given in [1] and [7]. Finally we present empirical results for all of the above procedures in sections 6 and 7 and make appropriate comparisons.

2. Notation, Definition and Requirement.

Let p_i denote the single-trial success for population π_i and let $q_i = 1 - p_i$ $(i = 1,2,\ldots, k)$. The ordered p-values are denoted by $p_{[1]} \leq p_{[2]} \leq \cdots \leq p_{[k]}$. For $p_{[k]} > p_{[k-1]}$, a correct selection (CS) is defined as the selection of the population associated with $p_{[k]}$; for

equality, either selection is correct. Let Δ denote the value of $p_{[k]} - p_{[k-1]}$. A procedure R is said to satisfy the (Δ^*, P^*)-probability (of a correct selection) requirement if

$$(2.1) \qquad P\{CS|R\} \geq P^* \text{ whenever } \Delta \geq \Delta^*;$$

here Δ^* (with $0 < \Delta^* < 1$) and P^* (with $\frac{1}{k} < P^* < 1$) are preassigned constants. All the procedures discussed in this paper satisfy this common requirement (2.1).

Let N_i denote the sample size taken from π_i and let N denote the sum over i of these sample sizes $(i = 1,2,\ldots, k)$. Let $N_{[i]}$ denote the sample size from the population associated with $p_{[i]}$ $(i = 1,2,\ldots, k)$. Then we define our loss function by

$$(2.2) \qquad L = \sum_{i=1}^{k} (p_{[k]} - p_{[i]})N_{[i]}$$

and the corresponding risk function by

$$(2.3) \qquad \text{Risk} = \sum_{i=1}^{k} (p_{[k]} - p_{[i]})E\{N_{[i]}\}.$$

In the applications that we have in mind, the one dealing with clinical trials is uppermost. In this application the primary concern is to reduce the use of poorer treatments. The risk function (2.3) represents the expected number of failures that could have been avoided if we had known beforehand which population is best.

In addition to the above risk function, we are also interested in reducing the expected total number of observations $E\{N|R\}$ for the procedure R.

3. Likelihood Procedure for Play-the-Winner Sampling.

In this section we consider the likelihood procedure that is appropriate for play-the-winner sampling. The case of general k is considered in Section 3.1 and this is specialized to $k = 2$ in Section 3.2.

3.1 Likelihood rule for PW sampling with general k.

A likelihood rule based on PW sampling and without early elimination can be developed in a manner similar to that given in [1]. We describe this procedure for $k = 3$ and specialize to $k = 2$ in section 3.2; the generalization to arbitrary k is a straightforward extension of the case $k = 3$.

Let $S_1 \leq S_2 \leq S_3$ denote the current number of successes from the 3 populations and let F_i represent the current number of failures from the population associated with S_i $(i = 1,2,3)$. If $S_3 = S_2$ $(> S_1)$ we associate S_3 with the smaller of the two corresponding F values; similarly for $S_1 = S_2 = S_3$ we assign S_3 to the one with the smallest F value. If it is still not determined then we use randomization. However it is shown below that for $P^* \geq \frac{1}{2}$ our rule never terminates sampling when randomization is used. Let $p_{[1]} \leq p_{[2]} \leq p_{[3]}$ be the ordered (unknown) probabilities of success on a single trial.

The method, based on results in [1], is to write the most likely of the 3 possible assignments of the pair (S_3, F_3) with the ordered p-values and to stop sampling when the minimum [over that part of the parameter space for which $p_{[3]} - p_{[2]} \geq \Delta^*$] of the corresponding likelihood ratio is at least P^*. More specifically, let the likelihood $L(\alpha, \beta, \gamma)$ be defined by

$$(3.1) \qquad L(\alpha, \beta, \gamma) = p_{[1]}^{S_\alpha}(1-p_{[1]})^{F_\alpha} p_{[2]}^{S_\beta}(1-p_{[2]})^{F_\beta} p_{[3]}^{S_\gamma}(1-p_{[3]})^{F_\gamma}$$

where (α, β, γ) is a permutation of $(1, 2, 3)$. Let the likelihood ratio $\mathfrak{L}(3)$ be defined by

$$(3.2) \qquad \mathcal{L}(3) = \frac{L(1,\,2,\,3) + L(2,\,1,\,3)}{\Sigma L(\alpha,\,\beta,\,\gamma)}$$

where the sum is over all $3! = 6$ possible permutations. This likelihood ratio $\mathcal{L}(3)$ associates S_3 with $p_{[3]}$ and it is proved in [1] (cf. ordering theorem in [1], page 66) that $\mathcal{L}(3) \geq \mathcal{L}(2) \geq \mathcal{L}(1)$, where $\mathcal{L}(j)$ is defined similarly to (3.2) and associates S_j with $p_{[3]}$ $(j = 1, 2)$. It is a basic result in [1] (pages 17, 18) that if any procedure R has the property at stopping time that

$$(3.3) \qquad \text{Min } \mathcal{L}(3) \geq P^*$$

(where the minimum is over all points in the parameter space for which $p_{[3]} - p_{[2]} \geq \Delta^*$) then R must satisfy the P^* condition (2.1). It is also shown in [1] (part 1 of theorem 6.1.1) that this minimum in (3.3) is attained at some (GLF) configuration in which

$$(3.4) \qquad p_{[1]} = p_{[2]} = p - \Delta^*$$

where we now use p to designate $p_{[3]}$. Hence by (3.2) and straightforward algebra we can write (3.3) in the form

$$(3.5) \qquad \underset{\Delta^* \leq p \leq 1}{\text{Max}} \left\{ \frac{L(3,2,1) + L(2,3,1) + L(1,3,2) + L(3,1,2)}{L(1,2,3) + L(2,1,3)} \right\} \leq \frac{1-P^*}{P^*} .$$

Using (3.1), we obtain for (3.5) the explicit form

$$(3.6) \qquad \underset{\Delta^* \leq p \leq 1}{\text{Max}} \left\{ \left(\frac{p-\Delta^*}{p} \right)^{T_1} \left(\frac{1-p}{1-p+\Delta^*} \right)^{U_1} + \left(\frac{p-\Delta^*}{p} \right)^{T_2} \left(\frac{1-p}{1-p+\Delta^*} \right)^{U_2} \right\} \leq \frac{1-P^*}{P^*} ,$$

where $T_i = S_3 - S_i$ and $U_i = F_i - F_3$ $(i = 1,2)$. For the case of VT sampling we note that $F_j - F_3 = S_3 - S_j$ $(j = 1,2)$ and the stopping rule

(3.6) reduces to that given in [1] and in section 7 below. For PW sampling the above does not hold and we note that $F_j - F_3$ can only take the values -1, 0 and +1. If $F_j - F_3 = -1$ for either $j = 1$ or $j = 2$ then the left side of (3.6) clearly tends to ∞ and hence the inequality cannot be satisfied. Thus we can state the

Stopping Rule for Procedure R_{LPWC}:

Stop sampling as soon as $F_3 \leq \text{Min}(F_1, F_2)$ and (3.6) holds.

The terminal decision rule is to select the population associated with S_3. If $S_3 = S_2$ and $F_3 = F_2$ at stopping time then we randomize between these two populations with probability 1/2 for each. Since $P^* > \frac{1}{3}$ we cannot terminate with both $S_3 = S_2 = S_1$ and $F_3 = F_2 = F_1$. Moreover, for $P^* \geq \frac{1}{2}$ the value of $(1-P^*)/P^* \leq 1$ and the inequality (3.6) cannot hold if $S_3 = S_2$ and $F_3 = F_2$. Hence we will never have to randomize in our termination rule when $P^* \geq \frac{1}{2}$.

It should be noted that the procedure R_{LPWC} is carried out by computing the maximum in (3.6) after every single observation and this can be tedious. However it is possible to use (3.6) to construct a set of stopping points for any given P^*. This set turns out to be fairly small and thus becomes a convenient method of describing the entire stopping rule. Illustrations of such stopping sets are given in Table 1A for $\Delta^* = .1, .2$ and $P^* = .75, .90, .95, .99$. For example, for the pair $(\Delta^* = .2, P^* = .90)$ there are 11 pairs of stopping points given in Table 1A. If $F_1 = F_2 = F_3$ then sampling terminates as soon as $T_2 = S_3 - S_2 \geq 10$ and $T_1 = S_3 - S_1 \geq 26$ or as soon as $T_2 \geq 11$ and $T_1 \geq 17$ or etc.

TABLE 1A

STOPPING POINTS FOR THE SEQUENTIAL LIKELIHOOD PROCEDURE

R_{LPWC} WITH k = 3 POPULATIONS

$\Delta^* = .1$

P^*	$F_1 = F_2 = F_3$ (T_1, T_2)[§]		$F_1 = F_2 = F_3+1$ (T_1, T_2)		$F_2 = F_3 = F_1-1$ (T_1, T_2)		$F_1 = F_3 = F_2-1$ (T_1, T_2)	
	18	17	11	10	13	13	13	13
	19	16	13	9	15	12	14	11
	20	15	14	8	20	11	15	10
.75	22	14	17	7			16	9
	25	13	23	6			18	8
	29	12					20	7
	38	11					26	6
	28	27	20	19	23	23	23	21
	30	26	21	18	24	22	24	20
	31	25	22	17	30	21	25	19
.90	33	24	24	16			26	18
	37	23	28	15			27	17
	42	22	36	14			29	16
	61	21					32	15
							40	14
	35	35	26	25	30	29	30	27
	36	34	27	24	37	28	31	26
	37	33	29	23			32	25
.95	38	32	31	22			33	24
	41	31	34	21			34	23
	44	30	42	20			36	22
	50	29					39	21
	78	28					47	20

$\Delta^* = .1$ (cont.)

.99	51	50	40	39	45	44	45	42
	52	49	41	38			46	40
	54	48	42	37			47	39
	56	47	44	36			48	38
	58	46	47	35			49	37
	63	45	53	34			51	36
	75	44					54	35
							59	34

$^{\S}T_1 = S_3 - S_1$, $T_2 = S_3 - S_2$ as defined in the text.

$\Delta^* = .2$

P^*	$F_1 = F_2 = F_3$ (T_1, T_2)		$F_1 = F_2 = F_3+1$ (T_1, T_2)		$F_2 = F_3 = F_1-1$ (T_1, T_2)		$F_1 = F_3 = F_2-1$ (T_1, T_2)	
.75	9	8	4	4	6	6	6	5
	10	7	6	3	9	5	7	4
	12	6	10	2			8	3
	24	5					11	2
.90	13	13	8	8	11	10	11	8
	15	12	9	7			12	7
	17	11	11	6			14	6
	26	10						
.95	17	16	11	10	14	14	14	11
	19	15	13	9			15	10
	22	14	20	8			17	9
							23	8

$\Delta^* = .2$ (cont.)

.99	24	24	17	17	21	21	21	18
	25	23	18	16			22	17
	27	22	20	15			23	16
	32	21					24	15

Note: Some numerical results for these procedures are given in Tables 3 and 4.

TABLE 1B

STOPPING POINTS FOR THE SEQUENTIAL LIKELIHOOD PROCEDURE

R_{LPWC} WITH $k = 2$ POPULATIONS

P^*	$\Delta^* = .1$		$\Delta^* = .2$	
	$F_1 = F_2$	$F_1 = F_2 + 1$	$F_1 = F_2$	$F_1 = F_2 + 1$
	$S_2 - S_1$	$S_2 - S_1$	$S_2 - S_1$	$S_2 - S_1$
.75	11	6	5	2
.90	21	14	10	6
.95	28	20	14	8
.99	44	34	21	15

A conservative variation of the above rule replaces $(1-p)/(1-p+\Delta^*)$ in (3.6) by its upper bound 1 when $F_3 \leq \text{Min}(F_1, F_2)$ and we obtain the

Stopping Rule for the Conservative Likelihood Procedure:

Stop sampling as soon as $F_3 \leq \text{Min}(F_1, F_2)$ and

$$(3.7) \qquad (1-\Delta^*)^{T_1} + (1-\Delta^*)^{T_2} \leq \frac{1-P^*}{P^*} .$$

It will be seen in section 7 that this conservative rule roughly causes a 20 percent increase in the total expected number of observations, E{N} above that for the procedure R_{LPWC}.

3.2 Likelihood rule for PW sampling with $k = 2$.

The special case $k = 2$ is of particular interest because we can make the procedure more explicit and because we can make comparisons with other procedures already studied, e.g., the procedure R_{PW} in [10]. The derivation in section 3.1 above gives for $k = 2$ the

Stopping Rule for R_{LPW} $(k = 2)$:

Stop as soon as $F_2 \leq F_1$ (i.e., $F_1 - F_2 = 0$ or 1) and

$$(3.8) \qquad \max_{\Delta^* \leq p \leq 1} \left\{ \left(\frac{p-\Delta^*}{p} \right)^{S_2 - S_1} \left(\frac{1-p}{1-p+\Delta^*} \right)^{F_1 - F_2} \right\} \leq \frac{1-P^*}{P^*} .$$

After randomization, let I denote the population that we sample from first and II the other population. Then $F_1 = F_2$ in (3.8) when we are sampling from I and $F_1 = F_2 + 1$ in (3.8) when we are sampling from II. Hence we stop and select I as soon as $S_2 - S_1 = t$ (if this happens before another equality below), where $t > 0$ is the smallest integer equal to or greater than the solution of

$$(3.9) \qquad t = \frac{\ln\{(1-P^*)/P^*\}}{\ln(1-\Delta^*)} .$$

We stop and select II as soon as $S_2 - S_1 = s$ (if this happens first) where $s > 0$ is the smallest integer for which

$$(3.10) \qquad \max_{\Delta^* \le p \le 1} \left\{ \left(\frac{p-\Delta^*}{p} \right)^s \left(\frac{1-p}{1-p+\Delta^*} \right) \right\} \le \frac{1 - P^*}{P^*} .$$

It is easily seen that $t \geq s$ and that we will only select a population after getting a success from that same population. This differs from the procedure R_{PW} in [10] only in that we allow $t \geq s$ and in [10] only $t = s$ is considered.

Using the recursion formula method given in [10] we can now derive an exact expression for the $P\{CS\}$, $E\{N\}$ and the expected number of observations $E\{N_B\}$ on the poorer treatment. This will be done for arbitrary positive s and t and, as a special case, we can then set s and t equal to the values obtained above by the likelihood approach. Let p (resp., p') be associated with population I (resp., II), let $NT = I$ mean that the next trial is on population I, let (s, t) denote the stopping points, and define

$$(3.11) \qquad \begin{aligned} P_n &= P_n(s, t) = P\{I \text{ is selected} \mid S_I - S_{II} = n, NT = I, (s, t)\} \\ Q_n &= Q_n(s, t) = P\{I \text{ is selected} \mid S_I - S_{II} = n, NT = II, (s, t)\}. \end{aligned}$$

Then the PW sampling scheme, conditional on I being the better population leads to the recursion

$$(3.12) \qquad \begin{aligned} P_n &= pP_{n+1} + qQ_n \\ Q_n &= p'Q_{n-1} + q'P_n \end{aligned}$$

with boundary conditions $P_t = 1$ and $Q_{-s} = 0$.

From (3.12) we find that

$$(3.13) \qquad P_n(s, t) = \frac{q' - q\lambda^{s+n}}{q' - q\lambda^{s+t}} \; ; \; Q_n(s, t) = \frac{q'(1-\lambda^{s+n})}{q' - q\lambda^{s+t}} \; ,$$

where $\lambda = p'/p \leq 1$. Setting $n = 0$, we obtain the conditional PCS $= P_0(s, t)$ given that I is the better population. For the same problem we define the dual expressions

$$(3.14) \qquad \begin{aligned} P'_n &= P'_n(s, t) = P\{\text{II is selected} \,|\, S_{II} - S_I = n, \text{NT} = \text{II}, (s, t)\}, \\ Q'_n &= Q'_n(s, t) = P\{\text{II is selected} \,|\, S_{II} - S_I = n, \text{NT} = \text{I}, (s, t)\}, \end{aligned}$$

and let p (resp., p') be associated with II (resp., I). Then we find that the recursive scheme is exactly as in (3.12) with the new boundary conditions $P'_s = 1$ and $Q'_{-t} = 0$, which differ from the above only in that s and t are interchanged. It follows that the conditional PCS given that II is the better population is $Q'_0(s, t)$ and this is obtained from (3.13) by merely interchanging s and t, i.e., $Q'_0(s, t) = Q_0(t, s)$. Hence, from these two conditional PCS results, we obtain

$$(3.15) \qquad P\{CS|R_{LPW}\} = \frac{P_0(s, t) + Q_Q(t, s)}{2} = \frac{q' - \frac{1}{2}(q\lambda^s + q'\lambda^t)}{q' - q\lambda^{s+t}} \; .$$

It is easily seen that for the three extreme cases $p' \to 0$, $p' \to p > 0$ and $p \to 1$ we obtain from (3.15) $P(CS) = 1, 1/2$ and $1 - \frac{1}{2}(p')^t$, respectively.

Using an analogous method to obtain $E\{N_B\}$, the number of observations on the poorer population, we define

$$U_n = U_n(s, t) = E\{N_{II} | S_I - S_{II} = n, NT = I, (s, t)\},$$

(3.16)

$$V_n = V_n(s, t) = E\{N_{II} | S_I - S_{II} = n, NT = II, (s, t)\}$$

and associate p with population I. Then the PW sampling scheme leads to the recursion

$$U_n = pU_{n+1} + qV_n$$

(3.17)

$$V_n = p'V_{n-1} + q'U_n + 1$$

with boundary conditions $U_t = 0 = V_{-s}$. It can be shown (and it is sufficient to verify) that

$$U_n(s, t) = \frac{q(t-n)}{p(1-\lambda)} - \frac{q[p + q(s+t)]\lambda^s(\lambda^n - \lambda^t)}{p(1-\lambda)(q' - q\lambda^{s+t})},$$

(3.18)

$$V_n(s, t) = \frac{p + q(t-n)}{p(1-\lambda)} - \frac{[p + q(s+t)]\lambda^s(q'\lambda^n - q\lambda^t)}{p(1-\lambda)(q' - q\lambda^{s+t})}.$$

The conditional $E\{N_B\}$ given that I is the better population is $V_0(s, t)$. We again define new quantities dual to (3.16) by writing

$$U_n' = U_n'(s, t) = E\{N_I | S_{II} - S_I = n, NT = II, (s, t)\}$$

(3.19)

$$V_n' = V_n'(s, t) = E\{N_I | S_{II} - S_I = n, NT = I, (s, t)\}$$

and letting p be associated with population II. Then we get the same recursion scheme as in (3.17) with the new boundary conditions $U_s' = 0 = V_{-t}'$, so that we need only interchange s and t in (3.18) to solve for (3.19). Hence the conditional $E\{N_B\}$ given that II is the better population is $V_0'(s, t) = V_0(t, s)$. Hence from (3.18)

$$(3.20)\qquad E\{N_B|R_{LPW}\} = \frac{U_0(s,\,t) + V_0(t,\,s)}{2} = \frac{[p + q(s+t)](1-\lambda^t)(q' - q\lambda^s)}{2p(1-\lambda)(q' - q\lambda^{s+t})}\,.$$

Similarly, to find $E\{N_A\}$ we replace N_{II} by N_I in (3.16) and obtain in place of (3.17)

$$(3.21)\qquad \begin{aligned} \tilde{U}_n &= p\tilde{U}_{n+1} + q\tilde{V}_n + 1 \\ \tilde{V}_n &= p'\tilde{V}_{n-1} + q'\tilde{U}_n \end{aligned}$$

with boundary conditions $\tilde{U}_t = 0 = \tilde{V}_{-s}$. The solution of this set is

$$(3.22)\qquad \begin{aligned} \tilde{U}_n(s,\,t) &= \frac{q'(t-n)}{p(1-\lambda)} - \frac{q[p' + q'(s+t)]\lambda^s(\lambda^n - \lambda^t)}{p(1-\lambda)(q' - q\lambda^{s+t})} \\ \tilde{V}_n(s,\,t) &= \frac{p' + q'(t-n)}{p(1-\lambda)} - \frac{[p' + q'(s+t)]\lambda^s(q'\lambda^n - q\lambda^t)}{p(1-\lambda)(q' - q\lambda^{s+t})}\,. \end{aligned}$$

Again we set up the dual quantities

$$(3.23)\qquad \begin{aligned} \tilde{U}'_n &= \tilde{U}'_n(s,\,t) = E\{N_{II}|S_{II} - S_I = n,\ NT = II,\ (s,\,t)\}, \\ \tilde{V}'_n &= \tilde{V}'_n(s,\,t) = E\{N_{II}|S_{II} - S_I = n,\ NT = I,\ (s,\,t)\} \end{aligned}$$

and let p be associated with population II. Then the recursion is the same as in (3.21) with the boundary conditions $\tilde{U}'_s = 0 = \tilde{V}'_{-t}$, so that we need only interchange s and t in (3.22). Hence the conditional $E\{N_A\}$ given that II is the better population is $\tilde{V}'_0(s,\,t) = \tilde{V}(t,\,s)$. It follows from (3.22) that

$$(3.24)\qquad E\{N_A|R_{LPW}\} = \frac{U_0(s,\,t) + V_0(t,\,s)}{2} = \frac{[p' + q'(s+t)](q' - q\lambda^s)(1-\lambda^t)}{2p(1-\lambda)(q' - q\lambda^{s+t})}\,.$$

Adding $E\{N_A\}$ and $E\{N_B\}$, gives

$$(3.25)\qquad E\{N|R_{LPW}\} = \left(\frac{\bar{p} + q(s+t)}{p}\right)\left(\frac{1 - \lambda^t}{1 - \lambda}\right)\left(\frac{q' - q\lambda^s}{q' - q\lambda^{s+t}}\right)$$

where $\bar{p} = (p+p')/2$ and $\bar{q} = 1 - \bar{p}$.

For the case $p = p'$, we take the limits in (3.20), (3.24) and (3.25) as $p' \to p$ and obtain

$$(3.26)\qquad E\{N_B|R_{LPW}\} = E\{N_A|R_{LPW}\} = \frac{1}{2} E\{N|R_{LPW}\} = \frac{t(p+qs)}{2p} ,$$

which is comparable with $r(p+qr)/2p$ obtained in (2.11) in [11] for the procedure R_{PW} with $s = t(=r)$. If, as is usually the case, we have $t > r > s$ and $st < r^2$, then each of these three expectations is smaller under R_{PW} for q close to zero and each is smaller under R_{LPW} for p close to zero. Thus neither of these procedures can be uniformly better than the other, i.e., throughout the parameter space. Since t in (3.9) is asymptotically ($\Delta^* \to 0$) like r in (2.13) of [11] it follows that the same lack of a uniform result holds in comparing R_{LPW} and the vector-at-a-time procedure R_{VT}, i.e., $E\{N|R_{VT}\}$ is smaller for $p \to 0$ and $E\{N|R_{LPW}\}$ is smaller for $p \to 1$.

To illustrate the results of procedure R_{LPW} and compare them with the procedure R_{PW} in [10] we consider the pair ($P^* = .95$, $\Delta^* = .2$) and put the results in tabular form. For the procedure R_{PW} we need to randomize between $r = 10$ (with probability .555) and $r = 11$ (with probability .445); this achieves the P^*- value .555(.945) + .445(.956) = .950 in the LF configuration. For the procedure R_{LPW} we randomize between the pair ($s = 7$, $t = 11$) with probability .434 and the pair ($s = 8$, $t = 12$) with probability .566; this achieves the P^*-value .434(.943)+.566(.955) = .950 in the LF configuration. In randomizing between these two particular pairs

(7, 11) and (8, 12) for procedure R_{LPW}, rather than other pairs, such as (7, 12) and (8, 12), our criterion was to minimize the maximum of $E\{N_B\}$, which generally occurs at $\bar{p} = \Delta^*/2$. This also seems to minimize the maximum for $E\{N\}$ and $E\{N_A\}$, which also generally occur at $\bar{p} = \Delta^*/2$.

TABLE 2

A COMPARISON OF PROCEDURES R_{PW} AND R_{LPW} FOR $k = 2$, $P^* = .95$, AND $\Delta^* = .2$ IN THE GLF CONFIGURATION $\Delta = .2$

$\bar{p} = \frac{p+p'}{2}$	$E\{N_B\}$		$E\{N_A\}$		$E\{N\}$	
	R_{PW}	R_{LPW}	R_{PW}	R_{LPW}	R_{PW}	R_{LPW}
.1	42.28	38.76	52.22	47.83	94.50	86.59
.2	37.31	34.22	47.25	43.29	84.55	77.51
.3	32.29	29.56	42.22	38.59	74.51	68.15
.4	27.13	24.71	36.99	33.61	64.12	58.33
.5	21.85	19.80	31.55	28.51	53.40	48.31
.6	16.60	15.04	26.08	23.50	42.68	38.54
.7	11.55	10.54	20.77	18.80	32.32	29.33
.8	6.77	6.33	15.79	14.50	22.56	20.83
.9	2.26	2.31	11.23	10.69	13.49	13.00

Note: Randomization was used to make $P^* = .95$ exactly in both cases--see text for details.

The comparison of R_{PW} and R_{LPW} in Table 2 shows that the latter has a smaller $E\{N_B\}$ and $E\{N\}$ in 17 out of the 18 entries. Thus we have effected a fairly uniform improvement, with emphasis on the maximum value at $\bar{p} = \Delta^*/2$, although (as was expected) the improvement is not substantial anywhere. However, in the context of clinical trials even slight decreases in $E\{N_B\}$ are important.

4. An Elimination Procedure R_{EPW}.

For $k > 2$ we define an elimination procedure which is an extension of the procedure R_{PW} defined for $k = 2$ and studied in [10]. Under R_{PW} we stop sampling when $|s_i - s_j| = r$ where s_i is the number of successes from π_i $(i \neq j;\ i = 1, 2;\ j = 1, 2)$. Assuming $s_i > s_j$, we then select π_i as the better population. The value of r required to satisfy (2.1) is the smallest integer equal to or greater than r_2, where

$$(4.1) \qquad r_2 = \frac{\ln 2(1-P^*)}{\ln (1-\Delta^*)} .$$

We extend this procedure as follows: Population π_j is eliminated if for some π_i (not yet eliminated) $s_i - s_j = r$. Let π_k be the best population. Since

$$(4.2) \qquad 1 - P\{CS\} \leq \Sigma P\{\pi_i \text{ eliminates } \pi_k\} \leq (k-1)(1-P^*),$$

it follows that

$$(4.3) \qquad P\{CS\} \geq 1 - (k-1)(1-P^*).$$

If we now set the right side of (3.29) equal to $\tilde{P}^*$, solve for P^*, and substitute the result in (3.27), then it is clear that the resulting procedure which uses throughout for r the smallest integer equal to or greater than

$$(4.4) \qquad r_k = \frac{\ln\left\{2\,\frac{(1-\tilde{P}^*)}{k-1}\right\}}{\ln(1-\Delta^*)}$$

satisfies

(4.5) $P\{CS|R_{EPW}\} \geq \tilde{P}^*$ whenever $\Delta \geq \Delta^*$.

Monte Carlo results for R_{EPW} and comparisons with other procedures can be found in Table 4.

5. Elimination with Wald's Double Dichotomy.

For $k \geq 2$ we investigate the numerical results of an elimination procedure which is derived in [5] and based on general methods from [3] applied to the double dichotomy problem as formulated by Wald [14].

This procedure R_{EVT} uses the VT sampling rule and eliminates population π_j if for some π_i (not yet eliminated)

(5.1) $s_i - s_j \geq c + dn^*$

where $c > 0$ and $d \geq 0$ are predetermined constants and n^* is the number of unlike pairs from π_i and π_j (i.e., observations in the same vector of the form S, F or F, S).

We now give the values of c and d that satisfy the requirement (2.1). Define τ_0 by

(5.2) $$\tau_0 = \left(\frac{1-\Delta^*}{1+\Delta^*}\right)^2 < 1$$

and let τ_1 denote any value such that

(5.3) $$\tau_0 < \tau_1 \leq 1/\tau_0 .$$

It is shown in [5] that by taking

(5.4) $$c = \frac{2 \ln\left(\frac{k-1}{1-P^*}\right)}{\ln(\tau_1/\tau_0)} , \quad d = \frac{2 \ln\left(\frac{1+\tau_1}{1+\tau_0}\right)}{\ln(\tau_1/\tau_0)} - 1$$

the requirement (2.1) will be satisfied. We select $\tau_1 = 1/\tau_0$ for our

Monte Carlo studies and this implies that $d = 0$, the reason being that asymptotically $(P^* \to 1)$ at the generalized least favorable configuration (i.e., when $p_{[1]} = p_{[2]} = \cdots = p_{[k-1]} = p_{[k]} - \Delta^*$) the risk defined in (2.3) is minimized for this value of τ_1 (cf. [5]).

The use of $d = 0$ above also provides us with the analogous elimination procedure for extending the procedure R_{VT} in [10] to $k > 2$ in the same way that we extended R_{PW} in section 4.

6. Comparisons of Several Play-the-Winner Procedures for $k = 2$.

In this section our aim is to make a comparison for $k = 2$ of the likelihood procedure developed in section 3.2 with some other procedures that satisfy the same probability requirement (2.1) with $\Delta^* = .2$ and $P^* = .95$. All of our numerical entries for $k = 2$ (in Table 3) are based on exact formulas. In Table 3 we have included in each cell $E\{N_B\}$ for the LF configuration $(\Delta = .2)$ and $E\{N\}$ for the LF $(\Delta = .2)$ and equal parameter (EP) configuration $(\Delta = 0)$. The procedures R_I and R_{IO} are inverse sampling procedures using PW sampling without and with reordering after each complete cycle, respectively.

The modified procedure R_H due to Hoel [3] uses PW-sampling and scores where the score W_A (for drug A, say) is defined by adding the successes of drug A and the failures of drug B and the termination rule is inverse sampling, i.e., stop when $\text{Max}(W_A, W_B) = r$. It has a bounded $E\{N\}$ value for $p = p' = 0$ and is therefore an improvement on R_I for small values of p.

Another procedure R_{IT} due to Berry and Sobel [2] modifies the inverse sampling scheme by terminating the procedure either after a fixed number c of complete cycles or after one population reaches r successes,

TABLE 3

EXPECTED SAMPLE SIZES FOR k = 2 UNDER FIVE PW-PROCEDURES ($P^* = .95$, $\Delta^* = .2$)

$\bar{p} = \frac{p+p'}{2}$	§	R_{LPW}	$R_I^{§§}$	$R_{IO}^{§§}$	$R_H^{\#}$	$R_{IT}^{\#\#}$
0	$E_0\{N\}$	∞	∞	∞	65.8	40.5
.1	$E\{N_B\}$	38.8	80.7	80.2	26.8	20.2
	$E\{N\}$	86.6	180.9	180.4	59.7	45.5
	$E_0\{N\}$	801.2	348.4	348.0	64.2	45.0
.2	$E\{N_B\}$	34.2	52.5	52.0	26.1	22.5
	$E\{N\}$	77.5	119.3	118.8	59.0	51.4
	$E_0\{N\}$	362.6	173.0	172.5	63.2	50.6
.3	$E\{N_B\}$	29.6	38.2	37.5	25.2	24.9
	$E\{N\}$	68.2	88.3	87.6	58.0	58.0
	$E_0\{N\}$	216.4	114.4	113.7	62.2	57.8
.4	$E\{N_B\}$	24.7	29.2	28.5	24.1	25.7
	$E\{N\}$	58.3	69.1	68.4	56.9	61.1
	$E_0\{N\}$	143.2	84.8	84.2	61.4	65.3
.5	$E\{N_B\}$	19.8	22.9	22.1	22.6	22.7
	$E\{N\}$	48.3	56.0	55.2	55.3	55.7
	$E_0\{N\}$	99.4	67.0	66.2	60.2	64.8
.6	$E\{N_B\}$	15.0	17.7	17.0	20.5	18.0
	$E\{N\}$	38.5	46.0	45.2	53.2	46.6
	$E_0\{N\}$	70.0	54.8	53.9	59.0	55.3
.7	$E\{N_B\}$	10.5	13.5	12.5	17.5	13.6
	$E\{N\}$	29.3	38.2	37.1	50.2	38.5
	$E_0\{N\}$	49.2	45.8	44.7	57.6	46.3
.8	$E\{N_B\}$	6.3	8.9	7.8	12.5	8.9
	$E\{N\}$	20.8	30.8	29.7	45.3	31.1
	$E_0\{N\}$	33.6	38.4	37.1	55.2	38.9
.9	$E\{N_B\}$	2.3	2.5	2.5	2.5	2.5
	$E\{N\}$	13.0	22.4	22.4	35.4	22.6
	$E_0\{N\}$	21.4	31.4	30.0	51.0	31.8
1.0	$E_0\{N\}$	11.6	20.0	20.0	33.4	20.2

§Giving $E\{N_B\}$, $E\{N\}$ for $\Delta = .2$ and $E_0\{N\}$ for $\Delta = 0$ in each cell.

§§Use r = 20 and 21 with weights .958 and .042, resp.; #use r = 33 and 34 with weights .6 and .4, resp.; ##use r = 20 and 21 with weights .761 and .239, resp.

whichever occurs sooner. This procedure appears to have two preassigned constants (r, c) to specify but both constants are used (with r = c) to satisfy (2.1).

Table 3 (for k = 2) shows that for $\bar{p} = (p+p')/2 > 1/2$ the likelihood procedure is preferable using either the risk criterion or E{N}. However, for $\bar{p} < 1/2$ the value of E{N} becomes infinite for all 3 of the procedures, R_{LPW}, R_I and R_{IO} when $p = p'$. Procedures R_H and R_{IT}, on the other hand, have a bounded E{N} function even for $p = p'$ and the numerical improvement for small $\bar{p}$ in Table 3, especially for $p = p'$, is very striking. It follows, as in the case of k = 3 in the next section, that if we had some a priori knowledge about the value of $\bar{p}$ we could more easily decide which of these procedures to use.

Procedure R_{IO} shows only a small improvement over procedure R_I but it is uniform over the entire parameter space.

7. Monte Carlo Simulation Studies for k = 3.

In this section we bring together several procedures appropriate for k = 3 populations and make some Monte Carlo studies to compare them. The criteria for comparison is the risk function (2.3) and the expected total number of observations E{N}. The same formulation (2.1) applies to all these procedures with the common values $P^* = .95$ and $\Delta^* = .2$. Each entry in Table 4A corresponds to the average of the results of 1000 experiments.

The main breakdown is between the procedures that use PW-sampling and those that use VT-sampling. We have included three previously published procedures. In the PW-group we include the inverse sampling procedure R_I studied in [13]. In the VT-group we include the procedure R_{BKS} which

was developed in [1] for general k, but details of which are given in [1, page 270] only for $k = 2$. We also include in the VT-group the procedure R_P due to Paulson [7]. A brief description of these procedures now follows.

Under procedure R_I we sample cyclically from 3 populations with PW sampling until any one of them has r successes; it is then selected to be the best population. The Monte Carlo results for R_I given in Table 4 are very close to approximate values based on (3.35) and (3.37) in [13]. A table of these approximate values, not included here, gives values consistently smaller than the observed values in Table 3.

Under procedure R_{BKS} we use vector sampling and stop as soon as

$$(7.1) \qquad \left(\frac{1-\Delta^*}{1+\Delta^*}\right)^{2T_1} + \left(\frac{1-\Delta^*}{1+\Delta^*}\right)^{2T_2} \leq \frac{1-P^*}{P^*}$$

and select the population associated with S_3. For $P^* > 1/2$ we will not stop when $S_3 = S_2$ and hence randomization will not be required at termination. It should be noted that the form of this procedure in (6.1) is similar to that of the conservative procedure in (3.7), but since the latter uses PW-sampling there is no direct comparability.

Under procedure R_P we take N_{ir} observations from population π_i $(i = 1,2,3)$, where N_{ir} is a Poisson random variable with mean J. Let s_{ir} (resp., f_{ir}) denote the total number of successes (resp., failures) from π_i up to and including the r^{th} stage. Then population π_j is eliminated at stage r if for some π_i (not yet eliminated) we have

$$(7.2) \qquad s_{jr} - f_{jr} \leq s_{ir} - f_{ir} + \frac{\ln \alpha}{\ln \lambda} + r\,A(\lambda)$$

where $\alpha = (1-P^*)/(k-1)$,

$$(7.3) \qquad A(\lambda) = J[\Delta^*(\lambda^2-1) - (\lambda-1)^2]/(\lambda \ln \lambda)$$

and λ is any value between 1 and $(1+\Delta^*)/(1-\Delta^*)$. For our Monte Carlo studies we use the same values for J and λ that were used in [7], namely $J = 1$ and $\lambda = (1 + .75\Delta^*)/(1- .75\Delta^*)$, which equals $23/17 = 1.353$ in our case. It should perhaps be remarked that the Poisson observations are not counted in computing $E\{N\}$ or the risk function.

Table 4A gives the empirical risk function, Table 4B gives the empirical $E\{N\}$ function, and Table 4C gives the estimated PCS function (or observed frequency of success).

As a group the PW-sampling procedures are different from the group of VT-sampling procedures. The latter procedures have remarkably constant risk and $E\{N\}$ for varying values of $\max p_i (i = 1, 2, 3)$ while the former procedures appear to be monotonically decreasing with $\max p_i$; the cross over point is about .65 in Table 4A and about .75 in Table 4B. It follows that if we had some prior knowledge about $\max p_i$ (only), we might be better able to decide which type of sampling to use.

Among the PW-sampling rules the procedures R_{EPW} and R_{LPW} are quite similar and uniformly better than both the procedures R_I and the conservative likelihood procedure. However, from Table 4B it appears that the procedure R_{LPW} is slightly better than R_{EPW}.

For the VT-sampling procedures the procedure R_{EVT} is preferable to both R_{BKS} and R_P using either the risk or the $E\{N\}$ criterion; the differences between the latter two procedures appears to be small.

In Table 4C we note, as expected, that all procedures satisfied the requirement (2.1) in all the experiments that were carried out. The PW-procedures, except for conservative likelihood, came closer to the nominal value $P^* = .95$ and hence were slightly more efficient. In addition, the columns of Table 4C give some indication of where the least favorable configuration is for the pair, $\Delta^* = .2$ and $P^* = .95$.

We wish to point out that we have not observed the expected number of stages required for termination since i) it is not clearly defined for all procedures and ii) it is not crucial for the application to clinical trials. It should also be pointed out that our Monte Carlo results are only for GLF configurations, where $p_{[1]} = p_{[2]} = p_{[3]} - .2$. In other configurations the elimination procedures are even more preferable, because non-competing populations can be eliminated early.

TABLE 4A

RISK FOR VARIOUS PROCEDURES $k = 3$, $\Delta^* = .2$ AND $P^* = .95$[§]

	Play-the-Winner Sampling				Vector Sampling		
	R_{EPW}	R_I		R_{LPW}		R_P	R_{EVT}
Max p_i	Sobel-Weiss (elimination)	Inverse Sampling	Likelihood Conservative	Likelihood	R_{BKS}	Paulson (elimination)	Wald's double dichotomy (elimination)
.20	22.82	45.47	27.57	22.83	9.95	10.14	9.95
.25	21.35	35.73	25.84	21.03	10.28	10.12	9.80
.30	19.67	29.08	23.99	20.05	10.29	10.01	9.61
.35	18.25	24.78	22.85	18.90	10.35	10.17	9.49
.40	17.25	21.34	21.58	17.60	10.25	9.95	9.44
.45	15.73	18.33	19.39	16.06	10.38	9.90	9.33
.50	14.72	16.26	18.18	14.95	10.40	10.14	9.21
.55	13.08	14.44	16.62	13.51	10.20	9.97	9.24
.60	12.02	12.63	14.61	11.89	10.23	10.01	9.32
.65	10.41	11.25	13.20	10.73	10.32	10.21	9.16
.70	8.71	9.85	11.40	9.22	10.44	10.07	9.18
.75	7.42	8.77	9.74	8.13	10.84	10.07	9.45
.80	6.13	7.41	7.85	6.36	10.70	9.95	9.52
.85	4.88	6.08	6.04	5.01	10.88	10.14	9.70
.90	3.62	4.71	4.36	3.48	10.79	10.19	9.72
.95	2.27	3.00	2.65	2.32	10.73	10.17	9.73
1.00	1.02	1.09	0.95	1.00	10.68	10.20	9.91

[§]GLF configurations with 1000 experiments per point.

TABLE 4B

EXPECTED TOTAL NUMBER OF OBSERVATIONS FOR VARIOUS PROCEDURES

$k = 3$, $\Delta^* = .2$ AND $P^* = .95$[§]

	Play-the-Winner Sampling				Vector Sampling		
Max p_i	R_{EPW} Sobel-Weiss (elimination)	R_I Inverse Sampling	Likelihood Conservative	R_{LPW} Likelihood	R_{BKS}	R_P Paulson (elimination)	R_{EVT} Wald's double dichotomy (elimination)
.20	184.7	368.5	223.3	184.8	74.6	81.1	74.6
.25	177.9	291.1	210.3	171.2	77.1	81.0	76.0
.30	166.5	238.4	196.6	164.1	77.2	80.6	75.8
.35	156.4	204.1	188.2	155.5	77.6	81.7	75.8
.40	149.6	176.9	178.8	145.9	76.9	79.8	75.8
.45	138.8	153.6	162.5	134.5	77.9	79.4	75.4
.50	131.6	137.3	153.5	126.1	78.0	81.2	74.6
.55	118.3	123.1	141.8	115.1	76.5	80.0	74.8
.60	109.2	109.6	126.8	103.0	76.7	80.2	75.2
.65	97.2	99.2	116.3	94.5	77.4	82.0	74.6
.70	84.5	89.0	102.9	83.0	78.3	80.8	74.8
.75	73.4	80.9	90.3	74.7	81.3	80.6	76.9
.80	62.3	71.8	75.9	61.4	80.2	80.2	77.4
.85	52.1	63.1	62.4	51.0	81.6	81.0	79.1
.90	42.9	54.3	49.6	39.3	80.9	81.5	78.9
.95	31.4	44.3	36.5	30.2	80.4	81.7	78.8
1.00	21.2	33.4	23.7	20.2	80.1	81.3	79.7

[§]GLF configurations with 1000 experiments per point.

TABLE 4C

PROBABILITY OF CORRECT SELECTION FOR VARIOUS PROCEDURES

$k = 3$, $\Delta^* = .2$ AND $P^* = .95$[§]

	Play-the-Winner Sampling				Vector Sampling		
	R_{EPW}	R_I		R_{LPW}		R_P	R_{EVT}
Max p_i	Sobel-Weiss (elimination)	Inverse Sampling	Likelihood Conservative	Likelihood	R_{BKS}	Paulson (elimination)	Wald's double dichotomy (elimination)
.20	1.000	1.000	1.000	1.000	1.000	.968	1.000
.25	1.000	1.000	1.000	1.000	1.000	.968	1.000
.30	1.000	1.000	1.000	1.000	.998	.966	.999
.35	1.000	1.000	1.000	1.000	.993	.973	.993
.40	1.000	.998	1.000	1.000	.984	.971	.993
.45	1.000	.991	1.000	.997	.984	.960	.989
.50	1.000	.984	1.000	.999	.980	.970	.982
.55	.996	.983	.997	.990	.957	.971	.974
.60	.991	.967	.997	.987	.961	.961	.970
.65	.994	.970	.997	.987	.964	.972	.977
.70	.991	.963	.995	.981	.973	.976	.969
.75	.972	.952	.990	.976	.983	.974	.981
.80	.970	.957	.985	.977	.983	.969	.989
.85	.967	.965	.987	.968	.994	.970	.997
.90	.968	.969	.981	.962	.999	.965	.997
.95	.952	.986	.978	.955	1.000	.966	1.000
1.00	.957	.996	.989	.969	1.000	.964	1.000

[§]GLF configurations with 1000 experiments per point.

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